MA 1118 - Multivariable Calculus
Final Exam - Quarter I - AY 02-03

Instructions: Work all problems. Read the problems carefully. Show appropriate work, as partial credit will be given. Two pages 8-1/2x11 notes and "Blue books" permitted. Scientific calculators **not** permitted.

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1. (25 points) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Clearly explain the reason(s) for your answer in each case:

a. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{n^2+5}{n^4+1}$

solution:

Observe that this is an alternating series, but also

$$a_n = (-1)^n \frac{n^2+5}{n^4+1} \qquad \Longrightarrow \qquad |a_n| = \frac{n^2+5}{n^4+1} \quad \to \quad \frac{n^2}{n^4} = \frac{1}{n^2}$$

for "large" n. Therefore, the given series converges absolutely by the limit comparison test and p-tests ($p = 2$).

b. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)\, e^n}{n!}$

solution:

For this series: $\quad a_n = \dfrac{\ln(n)e^n}{n!}$, and the behavior for "large" n is not obvious. Therefore, since we have both a variable exponent and a factorial, the ratio test is strongly suggested. (Note $a_n > 0$ for all n.) Proceeding

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{\frac{\ln(n+1)e^{n+1}}{(n+1)!}}{\frac{\ln(n)e^n}{n!}} = \frac{n!\,\ln(n+1)e^{n+1}}{(n+1)!\,\ln(n)e^n} = \frac{e\,\ln(n+1)}{(n+1)\,\ln(n)} =$$

and so

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e \quad \lim_{n\to\infty}\left[\frac{\ln(n+1)}{\ln(n)}\right] \quad \lim_{n\to\infty}\left[\frac{1}{(n+1)}\right]$$

$$= e \cdot \lim_{n\to\infty}\left[\frac{\frac{1}{n+1}}{\frac{1}{n}}\right] \cdot 0 = e \cdot 1 \cdot 0 = 0 < 1$$

solution:

where we used L'Hospital's rule on the logarithms limit. Therefore, by the ratio test, the original series converges (and absolutely because the terms were already positive).

c. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n+1)^2}$

solution:

For this series: $\quad a_n = \dfrac{n^2}{(n+1)^2}$. Therefore, for "large" n

$$a_n \to \frac{n^2}{n^2} = 1 \neq 0$$

Therefore, the original series diverges, since the terms don't go to zero.

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+4}$

solution:

For this series: $\quad a_n = (-1)^n \dfrac{n}{n^2+4}$. Therefore, for "large" n

$$a_n \to (-1)^n \frac{n}{n^2} = (-1)^n \frac{1}{n}$$

Therefore, the original series cannot converge absolutely since $|a_n|$ behaves like $1/n$, which diverges by the p-test (p=1). However, since the a_n alternate in sign, and approach zero uniformly as $n \to \infty$, then this series converges (conditionally) by the alternating series test.

2. (20 points) a. Use the Taylor polynomial (series) for $\sqrt{1+x}$, with the terms up through $n=2$, expanded around the point $x_0=0$ to approximate the value of $\sqrt{1.2}$.

solution:

Since we will need terms up through x^3 to estimate the error, we begin with the table

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$(1+x)^{1/2}$	1	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$	$\frac{-\frac{1}{4}}{2!}=-\frac{1}{8}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$	$\frac{\frac{3}{8}}{3!}=\frac{1}{16}$

Therefore, the Taylor series up through $n=2$ is

$$\sqrt{1+x} \doteq c_0+c_1x+c_2x^2 = 1+\frac{1}{2}\,x-\frac{1}{8}\,x^2$$

Hence, since $\sqrt{1+x}=\sqrt{1.2} \quad \Longrightarrow \quad x=0.2$, we then have

$$\sqrt{1.2} \doteq 1+\frac{1}{2}\,(.2)-\frac{1}{8}\,(.2)^2 = 1+.1-.005 = 1.095$$

(compared to an actual vaule of 1.095445...).

b. **Without actually computing** $\sqrt{1.2}$, estimate the error in this approximation.

solution:

According to the Taylor Remainder Theorem, the error in approximating any function $f(x)$ by a Taylor polynomial of degree n (i.e. the first n terms of its Taylor series) is:

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-x_0)^{n+1}$$

where ξ is some undetermined point in the interval between x_0 and x.

solution:

Moreover, the error can also be estimated, generally to at least the correct order of magnitude, by evaluting the remainder expression at $\xi = x_0$ (i.e., in this case, at $\xi = 0$). This produces

$$R_2(.2) = \frac{f^{(3)}(0)}{(3)!}(.2)^3 = c_3(.2)^3 = \frac{1}{16}(.008) = .0005$$

a value which compares very favorably with the true error of 0.000445....

3. (20 points) a. Find an equation for the tangent line to the curve:

$$\mathbf{r}(t) \;=\; \cos(t)\ \mathbf{i} \;+\; \ln(1+2t)\ \mathbf{j} \;+\; t\sin(3t)\ \mathbf{k} \qquad \text{at} \qquad t=\frac{\pi}{2}$$

solution:

To write the equation for a line in space, we need to know a point on the line and the direction of the line. But we also know that the derivative of $\mathbf{r}(t)$, i.e. $\dfrac{d\mathbf{r}}{dt}$ represents the velocity vector for the motion described by $\mathbf{r}(t)$, and therefore lies in the direction of the tangent to the trajectory traced out by $\mathbf{r}(t)$. But

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = -\sin(t)\mathbf{i} + \frac{2}{1+2t}\mathbf{j} + (\sin(3t) + 3t\cos(3t))\,\mathbf{k}$$

Therefore, at $t = \pi/2$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\sin(\pi/2)\ \mathbf{i} + \frac{2}{1+2(\pi/2)}\ \mathbf{j} + \big(\sin(3(\pi/2)) + 3(\pi/2)\cos(3(\pi/2))\big)\ \mathbf{k}$$

$$= -(1)\ \mathbf{i} + \frac{2}{1+\pi}\ \mathbf{j} + (-1)\ \mathbf{k} = -\mathbf{i} + \frac{2}{1+\pi}\ \mathbf{j} - \mathbf{k}$$

Since the tangent vector must pass through the point on the curve at $t = \pi/2$, we find the point as

$$\mathbf{r}(\pi/2) \;=\; \cos(\pi/2)\ \mathbf{i} \;+\; \ln(1+2(\pi/2))\ \mathbf{j} \;+\; (\pi/2)\sin(3\pi/2)\ \mathbf{k}$$

$$= (0)\ \mathbf{i} \;+\; \ln(1+\pi)\ \mathbf{j} + (\pi/2)(-1)\ \mathbf{k} = \ln(1+\pi)\ \mathbf{j} - \frac{\pi}{2}\ \mathbf{k}$$

and so the equation of the tangent line is

$$\ln(1+\pi)\ \mathbf{j} - \frac{\pi}{2}\ \mathbf{k} \;+\; s\left(-\mathbf{i} + \frac{2}{1+\pi}\ \mathbf{j} - \mathbf{k}\right)$$

or

$$x = -s \quad , \quad y = \ln(1+\pi) + \frac{2}{1+\pi}\,s \quad , \quad z = -\frac{\pi}{2} - s$$

b. Find an equation for the plane through the point $\mathbf{P}_0 = (-1, 1, 10)$ and tangent to the surface

$$6x^2 - 4y^2 = 12 - z$$

solution:

The tangent plane to a surface $z = f(x, y)$ is, of course, simply the linearization of that surface at that point, i.e. the function

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

In this case, the original surface can be written

$$z = 12 - 6x^2 + 4y^2 \equiv f(x, y) \qquad \Longrightarrow \qquad \begin{aligned} f_x &= -12x \\ f_y &= 8y \end{aligned}$$

and so the linearization is

$$z = 10 + (-12(-1))(x - (-1)) + (8(1))(y - 1)$$

or

$$z = 14 + 12x + 8y$$

We could also find this by computing the gradient to level surface

$$z + 6x^2 - 4y^2 = 12$$

at the point $(-1, 1, 10)$, and then, using that as the normal to the plane, write the standard equation.

4. (25 points) Given

$$f(u,v) = uv^2 + \sin(uv) \quad , \quad \text{and} \quad u = x^2 y \ \ , \ v = xe^{xy}$$

find $\quad \dfrac{\partial f}{\partial x} \ , \quad \dfrac{\partial^2 f}{\partial u^2} \quad$ and $\quad \dfrac{\partial^2 f}{\partial v \partial u}$

solution:

According to the chain rule for partial derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\
&= \big(v^2 + v\cos(uv)\big)(2xy) + \big(2uv + u\cos(uv)\big)\left(e^{xy} + xye^{xy}\right) \\
&= \Big(\left(xe^{xy}\right)^2 + \left(xe^{xy}\right)\cos\!\big(\left(x^2y\right)\left(xe^{xy}\right)\big)\Big)(2xy) \ + \\
&\qquad \Big(2\left(x^2y\right)\left(xe^{xy}\right) + \left(x^2y\right)\cos\!\big(\left(x^2y\right)\left(xe^{xy}\right)\big)\Big)\left(e^{xy} + xye^{xy}\right)
\end{aligned}$$

Similarly

$$\frac{\partial f}{\partial u} = v^2 + v\cos(uv)$$

and so

$$\frac{\partial^2 f}{\partial u^2} = -v^2\sin(uv)$$

and

$$\frac{\partial^2 f}{\partial v \partial u} = 2v + \cos(uv) - uv\sin(uv)$$

5. (30 points) Find and correctly identify all the local maxima, minima and saddle points of

$$f(x, y) = 2y^2 - 2y^3 + 4xy - x^2$$

solution:

The critical points for this function are given by

$$f_x = 4y - 2x = 0 \qquad \Longrightarrow \qquad x = 2y$$

$$f_y = 4y - 6y^2 + 4x = 0$$

Substituting the expression for x in terms of y into the second equation yields

$$f_y = 4y - 6y^2 + 8y = 12y - 6y^2 = 6y(2 - y) = 0 \qquad \Longrightarrow \qquad y = 0,\ 2$$

Therefore, since $x = 2y$, there are only two critical points, $(0, 0)$ and $(4, 2)$. Setting up the standard table, we have

x	y	$f_{xx} = -2$	$f_{yy} = 4 - 12y$	$f_{xy} = 4$	$f_{xx}f_{yy} - f_{xy}^2$	Type
0	0	-2	4	4	$(-2)(4) - (4)^2 = -24$	saddle
4	2	-2	-20	4	$(-2)(-20) - (4)^2 = 24 > 0$	max

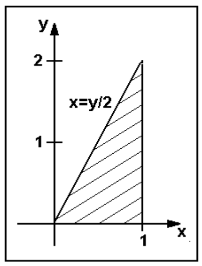
where $(4, 2)$ must be a local maxima because $f_{xx} < 0$ there. (Or, alternatively, because $f_{yy} < 0$ there.)

6. (35 points) Interchange the order of integration in the following integral and then compute its value:

$$\int_0^2 \int_{y/2}^1 e^{x^2}\ dxdy$$

solution:

The associated region of integration is the shaded area shown below:

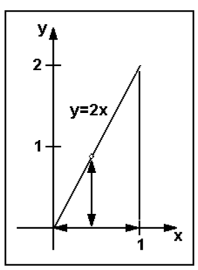


This same region can be "covered" by

(1) Letting x take on every value between zero and one

(2) At each of these values of x, letting y take on every value between $y = 0$ and $y = 2x$.

i.e. by:



Therefore, the integral can be rewritten

$$\int_{x=0}^{1} \Bigg\{ \underbrace{\int_{y=0}^{2x} e^{x^2} dy}_{ye^{x^2}\Big|_{y=0}^{2x}} \Bigg\} dx = \int_{x=0}^{1} 2xe^{x^2}\ dx = e^{x^2}\Big|_{x=0}^{1} = e - 1$$

7. (35 points) Consider the solid region R in space which lies above the plane $z = -1$, and *inside* the sphere

$$x^2 + y^2 + z^2 = 4 \ .$$

Set up **(DO NOT EVALUATE)** the *iterated* integrals needed to find

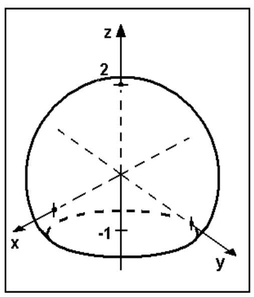
$$\iiint\limits_R x \, dV$$

in each of the following coordinate systems:

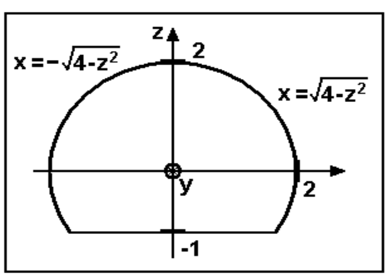
(a) Cartesian.
(b) Cylindrical.
(c) Spherical.

solution:

The associated region of integration is shown below:



To integrate this in Cartesian coordinates, the best "shadow" to use is in either the xz or yz planes, since the "shadow" in the xy plane includes a interior "edge" where the plane and sphere interest. The " shadow" in the xz plane is



This region can be "covered" by

(1) For every value of z between -1 and 2,
(2) Letting x vary from $-\sqrt{4-z^2}$ to $+\sqrt{4-z^2}$.

solution:

Then, at every point in that "shadow," a line normal to the xz plane will intersect the surface of the solid exactly twice

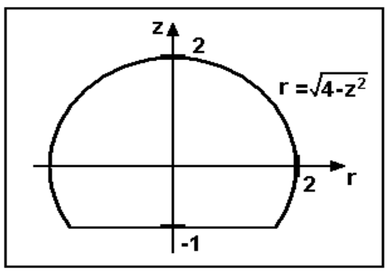
(1) Entering at $y = -\sqrt{4-x^2-z^2}$, and

(2) Exiting at $y = +\sqrt{4-x^2-z^2}$.

Therefore, the requisite integral is

$$\int_{z=-1}^{2} \int_{x=-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{y=-\sqrt{4-x^2-z^2}}^{\sqrt{4-x^2-z^2}} x \; dy \; dx \; dz$$

In cylindrical coordinates, we can use essentially the same figure as the "side" view,



and, since, because of symmetry, θ must take on every value between zero and 2π, the integral becomes (replacing the x in the intgerand by its value in terms of cylindrical coordinates):

$$\int_{z=-1}^{2} \int_{r=0}^{\sqrt{4-z^2}} \int_{\theta=0}^{2\pi} (r\cos(\theta)) \; r \; d\theta \; dr \; dz$$

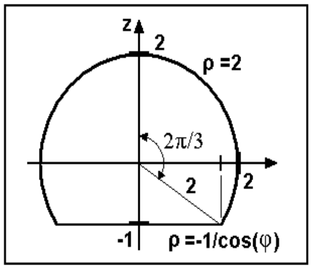
$$= \int_{z=-1}^{2} \int_{r=0}^{\sqrt{4-z^2}} \int_{\theta=0}^{2\pi} r^2 \cos(\theta) \; d\theta \; dr \; dz$$

The situation in spherical coordinates is, unfortunately, a bit "nastier," since the formula for the value or ρ on the boundary changes depending on ϕ. Specifically,

$$\begin{array}{lll} \rho & = 2 & , \quad 0 \le \phi \le 2\pi/3 \\ \rho & = -1/\cos(\phi) & , \quad 2\pi/3 < \phi \le \pi \end{array}$$

(see next figure).

solution:



Therefore, unfortunately, in this case, we must express the integral as a sum of two integrals. Specifically (again replacing the integrand of x by its value in terms of spherical coordintes:

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi/3}\int_{\rho=0}^{2} (\rho\sin(\phi)\cos(\theta))\,\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$$
$$+\int_{\theta=0}^{2\pi}\int_{\phi=2\pi/3}^{\pi}\int_{\rho=0}^{-1/\cos(\phi)} (\rho\sin(\phi)\cos(\theta))\,\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$$

or

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi/3}\int_{\rho=0}^{2} \rho^3\sin^2(\phi)\cos(\theta)\ d\rho\ d\phi\ d\theta$$
$$+\int_{\theta=0}^{2\pi}\int_{\phi=2\pi/3}^{\pi}\int_{\rho=0}^{-1/\cos(\phi)} \rho^3\sin^2(\phi)\cos(\theta)\ d\rho\ d\phi\ d\theta$$

(Whew!!!!)

8. (10 points) Find: $\displaystyle\sum_{n=1}^{\infty} 3^n x^{2n}$

solution:

This series can be rewritten:

$$\sum_{n=1}^{\infty} 3^n x^{2n} = \sum_{n=1}^{\infty} \left(3x^2\right)^n$$

which is essentially a geometric series ($r = 3x^2$), except that the geometric series starts with $n = 0$, not $n = 1$. That, however, is easily fixed, i.e.

$$\sum_{n=1}^{\infty} \left(3x^2\right)^n = \sum_{n=0}^{\infty} \left(3x^2\right)^n - 1 = \frac{1}{1-3x^2} - 1 = \frac{3x^2}{1-3x^2}$$